Computational Geometry Column 32

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Abstract

The proof of Dey's new k-set bound is illustrated.

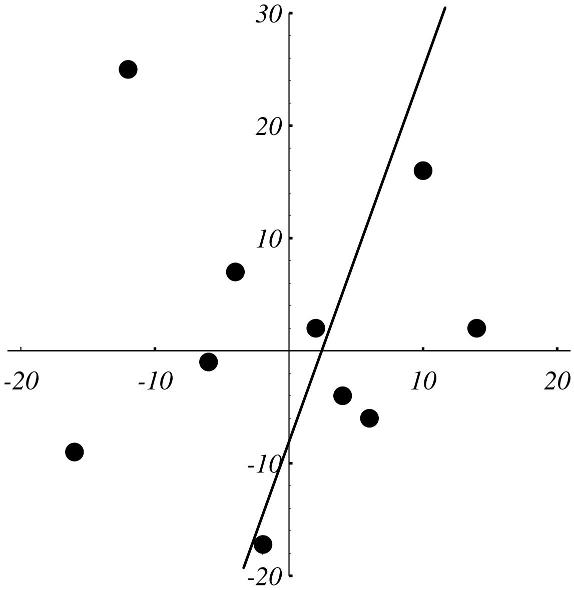

Figure 1: The line shown determines two 5-sets. There are a total of twenty-four 5-sets in this set of $n = 10$ points.

A *k-set* of a set of n points P in the plane is a set of exactly k points of P contained in an open halfplane. The maximum number of k-sets as a function of n and $k > 0$ is an important quantity, but a gap in the known bounds between $\Omega(n \log k)$ and $O(nk^{1/2})$ stood for over 25 years. The only progress was a tight bound of nk on the number of $\leq k$-sets, and a slight lowering of the upper bound for k-sets to $O(nk^{1/2}/\log^* k)$.[1] But recently, building

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[1]The first result, which will play a role below, is due to Alon and Győri [AG86]; the second is due to Pach et al. [PSS92]. See [Ede87] and [AAS97] for history and further references.

on the work of Agarwal, Aronov, and Sharir [AAS97], Tamal Dey improved the upper bound to $O(nk^{1/3})$ [Dey97]. Here I will sketch his proof[2] on an example.

Consider the set of $n = 10$ points shown in Fig. 1.[3] The line shown determines a 5-set, as there are exactly 5 points in the halfplane below. In this instance, there are also 5 points above, so this line determines two 5-sets. We will only count the k-sets below their halfplane bounding line, which clearly suffices for an asymptotic bound; this will be our first reduction.

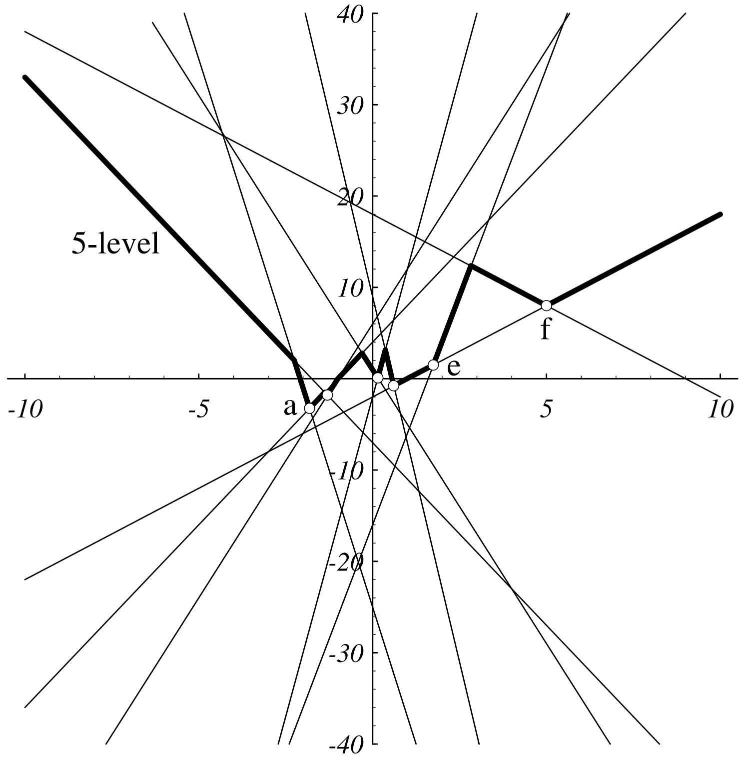

Figure 2: The arrangement dual to the points in Fig. 1, with the 5-level highlighted. The six vertices of V_4 ($a, \ldots, f$, left to right) are circled. The other vertices of the 5-level belong to V_5.

The next step is familiar to those working in combinatorial geometry: dualize each point of P to a line, transforming the point set to an arrangement of lines $\mathcal{A}$. I will use the transformation $(a, b) \mapsto y = ax - b$ from primal to dual (and $(a', b') \mapsto y = a'x - b'$ from dual to primal), which preserves the above-below relation: a point of P is below a line L iff the dual line is below the point dual to L. Define the k-level of $\mathcal{A}$ to be the closure of the set of points of the lines of $\mathcal{A}$ that have exactly k lines below them. The number of vertices on the k-level and the number of k-sets (with k points below) differ by at most 1. Take any k-set line, translate it upward until it hits a point of P, and then rotate it counterclockwise about

[2] I will follow his first proof, which employs duality. He has since developed a related proof that operates entirely in the primal setting.

[3] I will assume throughout that no three points of P lie on a line, and no two points lie on a vertical. Neither assumption affects the worst-case bounds.

that point until it hits another. This line corresponds to a vertex of the k-level. Fig. 2 shows the 5-level of the arrangement dual to the points in Fig. 1. Note that not every point of the 5-level has five lines below: for example, the rightmost vertex of the level, at $f = (5, 8)$, has four lines below.[4] Let V_k be the set of vertices of $\mathcal{A}$ with exactly k lines below. Then the vertices of the k-level are precisely $V_k \cup V_{k-1}$. Thus the vertex at f is a member of V_4; all six vertices in V_4 are marked with open circles in the figure.

The second reduction is that it suffices to bound $|V_{k-1}|$, as these vertices correspond to lines through two points of P that have exactly $k-1$ points below them. Perturbing the line corresponding to a vertex of V_{k-1} results in two k-sets below. Thus the point $f = (5, 8)$ in Fig. 2 maps to the line $y = 5x - 8$ through points $(-1, -18)$ and $(2, 2)$ of P; the perturbation that includes the former point but excludes the latter is the line shown in Fig. 1.

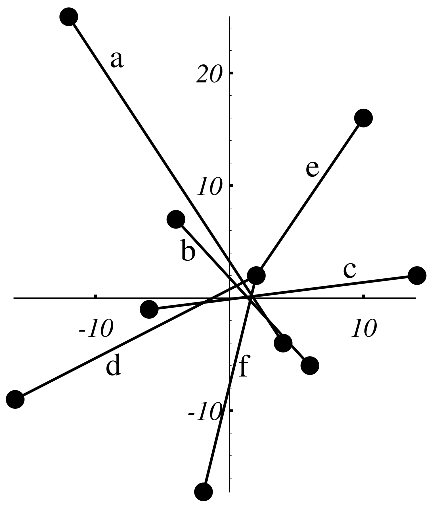

Figure 3: The graph G has an edge for every vertex of V_{k-1}. The edge labels match those in Fig. 2.

Now Dey defines a straight-line geometric graph G in the primal setting that connects a pair of points of P with an edge iff the line containing that edge dualizes to a vertex in V_{k-1}. See Fig. 3. We now seek to bound the number of edges $t = |V_{k-1}|$ of G. Dey obtained his bound by relating t to the number of crossings X of edges of G. In particular, Ajtai et al. [ACNS82] and Leighton [Lei84] proved that there must be at least ct^3/n^2 crossings among $t > 4n$ edges of a straight-line geometric graph, c a constant. As the case $t \leq 4n$ is easy,[5] this establishes that $ct^3/n^2 < X$.

An upper bound for X is obtained by a close analysis of the lines below the k-level, which I will only sketch. Agarwal et al. emphasized in [AAS97] a way of viewing those lines that is a key part of Dey's argument. They partitioned the portion of $\mathcal{A}$ below the k-level into a union of k concave chains $c_1, c_2, \cdots$, which turn only at the vertices of V_{k-1}. Fig. 4 will serve in place of a formal definition. A consequence of our duality is that a pair of crossing edges

[4] This is why the level was defined via closure, to fill in these holes.

[5] Our running example only has $t = 6 < 4n = 40$.

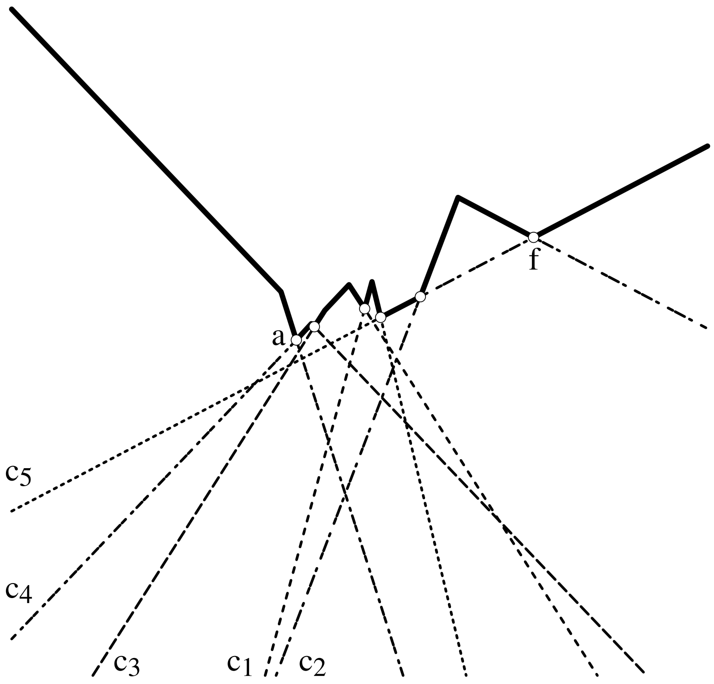

Figure 4: The $k = 5$ concave chains cover the arrangement below the k-level.

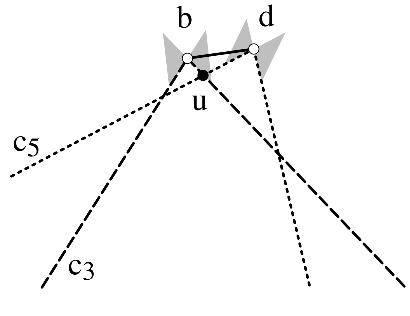

Figure 5: The common tangent to chains c_3 at vertex b and c_5 at vertex d (corresponding to the crossing of edges b and d in G (Fig. 3)) is charged to the point u where these two chains cross below the tangent. Since the common tangents of a pair of x-monotone concave chains have disjoint x-spans, this charging is unique.

in G corresponds to a line tangent to two concave chains, passing through a vertex of each; see Fig. 5. This bounds X by the number of such common tangents. The number of these tangents is in turn bounded from above by the number of times the chains cross each other, as explained in the caption to Fig. 5. As all of these crossings occur below the k-level of $\mathcal{A}$, there cannot be more than the number of vertices below this level which, as mentioned previously, is known to be at most nk.

So we now have $ct^3/n^2 < X < nk$. Solving for t yields $t = O(nk^{1/3})$.

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