Computational Geometry Column 31

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Abstract

Several topics related to packing on a sphere are discussed: packing points, packing lines through the center, and packing k-flats in dimension d.

Two frequently asked questions in the graphics community (e.g., on the Usenet newsgroup `comp.graphics.algorithms`) are: How can points be randomly placed on a sphere?, and, How can points be regularly arranged on a sphere? I will start with these two questions and move to generalizations: packing lines in space and packing affine subspaces in higher-dimensional spaces.

1 Random Points on a Sphere

Define a *zone* on a sphere to be the portion between two parallel planes that both intersect the sphere. For a sphere S of radius r, the area of a zone of width h is $2\pi rh$. This area is independent of where S is sliced, a fact that may seem countertuitive. The implication is that the z-coordinates (say) of random points on a sphere are uniformly distributed. Thus random points (x, y, z) on a sphere of radius 1 can be generated as follows:

1. Choose $z \in [-1, 1]$ uniformly.
2. Choose $\theta \in [0, 2\pi)$ uniformly.
3. Let $r = \sqrt{1 - z^2}$. Set $x = r\cos\theta$ and $y = r\sin\theta$.

See [Arv91], [Sho92], and [FAQ97, Subj. 6.08].

2 Packing Points on a Sphere

Often more regular arrangements of points are desired, which leads to the topic of packing points on a sphere; such arrangements are sometimes called "spherical codes" [CS93]. The goal is to distribute on the surface n points as evenly as possible, for example, maximizing the angular separation of any two points. One easy route to good, but usually not optimal packings, is to subdivide the inscribed icosahedron: place a point at the barycenter

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of every face, and continue this process recursively. Optimal or near-optimal packings are found by optimization techniques. One technique simulates the electrostatic repulsion of n electrons confined to the sphere surface [FAQ97, Subj. 6.06]. More sophisticated search techniques [HS93] have resulted in stunning packings, such as the one shown in Fig. 2. This packing is significantly more uniform than that obtained by subdividing an icosahedron.

3 Packing Lines Through the Center of a Sphere

The problem of radiating tumors with maximally-separated high-energy beams led Conway, Hardin, and Sloane [CHS96] to modify the same optimizing software that produced Fig. 2 to search for packings of lines through the origin, i.e., packing points on a sphere where all points come in antipodal pairs. For example, for $n = 6$, the best packing is determined (not surprisingly) by the diameters of the regular icosahedron (Fig. 1); for $n = 5$, the best packing (less obviously) is five of these six diameters. Both of these results were proven optimal by Fejes Tóth thirty years ago, but most packings reported in [CHS96] are not known to be optimal.

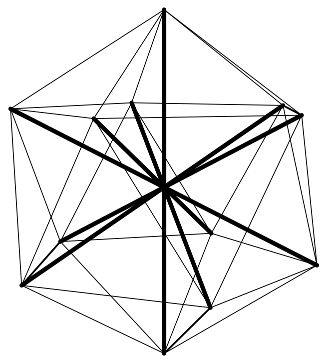

Figure 1: Six diameters of a regular icosahedron.

4 Packings in Grassmannian Spaces

The more general problem is to pack k-flats in d-dimensional space. The space of all k-dimensional subspaces of $\mathbb{R}^d$ is known as the Grassmannian manifold $G(d, k)$. Packing lines in 3-space is then packing in $G(3, 1)$. Packings in $G(d, 2)$ have an application suggested by Asimov [Asi85]: multidimensional data can be explored by projecting it to two-dimensional subspaces. Ideally one wants these subspaces "maximally separated" in some sense. The sense chosen in [CHS96] depends on what they call the *chordal distance* between subspaces. Some of the packings can be proved optimal with respect to this metric; for example, their packing of 18 planes in $\mathbb{R}^4$ is optimal. Generalizing to higher dimensions, they found in $G(8, 4)$ an optimal packing of 70 4-spaces in $\mathbb{R}^8$.

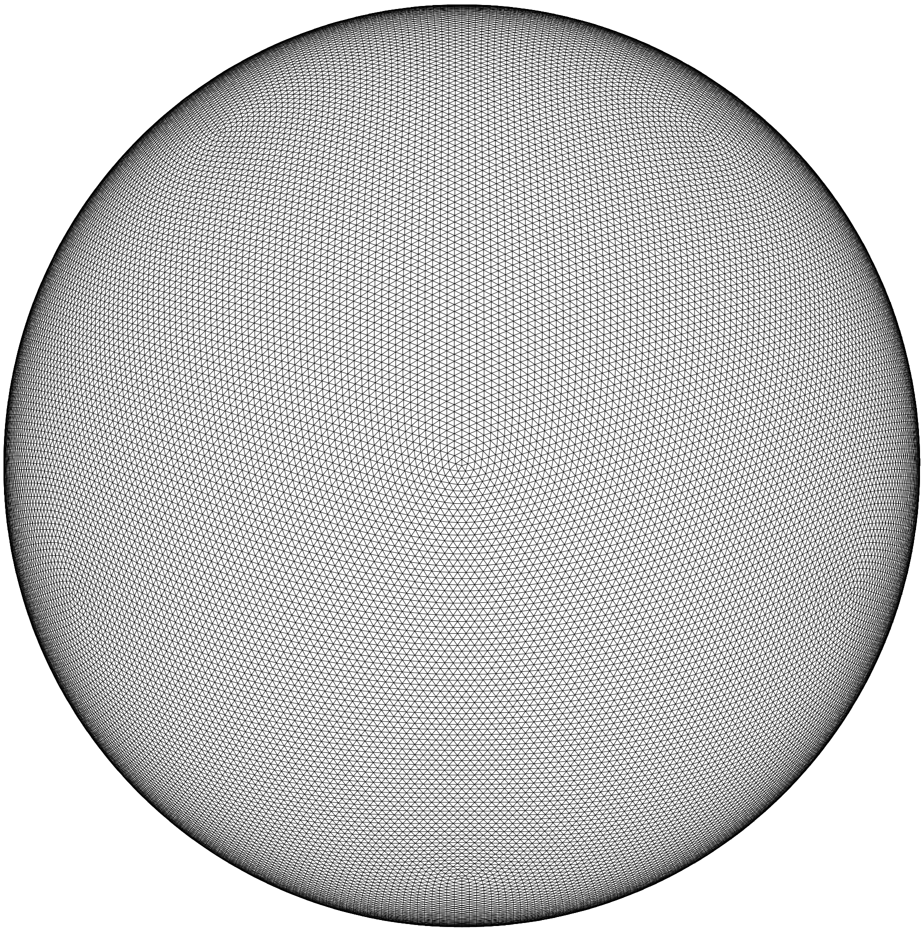

Figure 2: 48,002 points on a sphere. The angular separation between points is about 1°. All except 12 points have 6 neighbors; the remaining 12, which are the vertices of an icosahedron, have 5 neighbors. Copyright R. H. Hardin and N. J. A. Sloane, with permission.

Optimal packings are hard to come by, however. It was thus a considerable surprise when work in quantum coding theory led to optimal packings in $G(d, d/2)$ for any d a power of 2. Peter Shor's work on quantum computation produced a group equivalent to the 8-dimensional symmetry group of the optimal $G(8, 4)$ packing mentioned above. Pursing this coincidence, Shor and Sloane constructed optimal packings of $(d-1)(d+2)$ subspaces in $G(d, d/2)$, generalizing the 18 plane and 70 4-space packings to arbitary dimensions $d = 2^m$ [SS97].

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