Computational Geometry Column 30

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Abstract

Several results from *Combinatorial Geometry* [PA95] are detailed.

Below are listed a number of results (and a few open problems) from the text *Combinatorial Geometry*, by János Pach and Pankaj K. Agarwal [PA95]. The results were selected to indicate the flavor and variety of questions and answers in this rich field, which draws on number theory, classical geometry, combinatorics, and computational geometry.

1. A convex polytope of n vertices inscribed in a d-dimensional sphere has volume at most $n/2^d$ that of the sphere [Thm. 2.11 (Elekes), p. 16]. This result implies that there is no polynomial-time algorithm using a point-query oracle for estimating the volume of a convex body in d-space up to a constant factor.

2. Any packing of congruent disks covers at most $\pi/\sqrt{12} \approx 0.907$ of the plane, the density achieved by the lattice "penny packing" [Thue's Thm., p. 21]. The "cannonball packing" covers $\pi/\sqrt{18} \approx 0.740$ of 3-space with congruent balls, but whether this has been proven optimal remains controversial [p. 55].

3. Which convex shape has the smallest packing density in the plane? Here the goal is to find a shape that packs poorly, over all packings. It is not known whether there is a shape whose packing density in this sense is less than ≈ 0.902, achieved with a "smoothed octagon" [pp. 30, 41].

4. Any planar graph G can be realized by a packing of circular disks in the plane, each disk representing a node of G, with two disks touching iff there is a corresponding edge in G [Thm. 8.1 (Koebe), p. 96].

5. Let G be a connected graph with $n \geq 3$ vertices, such that for every pair of non-adjacent vertices x and y, $\deg(x) + \deg(y) \geq n$. Then G has a Hamiltonian cycle [Thm. 9.1(i) (Dirac), p. 118.]

6. Given any $k \geq 3$, there is an n such that any n points in the plane (no three collinear) contains the vertices of a convex k-gon [Thm. 9.20, Erdős, Szekeres, p. 135].

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7. The maximum number of times the unit distance can be realized by n points in the plane is $O(n^{4/3})$ [Cor. 11.11 (Spencer, Szemerédi, Trotter), p. 179]. The maximum in 3-space is $O(n^{5/3})$ [Thm. 10.8 (Erdős), p. 147]. Although for $d \geq 4$, the exact behavior of these repeated distance maxima are known, the two quoted results are not believed to be tight. The lower bound is $\Omega(n^{4/3} \log\log n)$ in 3-space, and $\Omega(n^{1+c/\log\log n})$ in the plane [Thm. 10.4 (Erdős), p. 144].

8. The maximum number of incidences between m points and n lines in the plane is $O(m^{2/3}n^{2/3} + m + n)$ [Thm. 11.8 (Szemerédi, Trotter), p. 176]. This result is closely related to the $O(n^{4/3})$ bound on repeated distances in the plane [O'R90].

9. If n points in the plane are connected by e line segments in such a way that no two segments *cross*, then $e \leq 3n - 6$ [Prop. 14.11, p. 230]. Two edges cross if they share an interior point. This result simply reflects the maximum number of edges of a plane graph on n vertices. If n points in the plane are connected by e line segments in such a way that no two segments are *disjoint*, then $e \leq n$ [Thm. 14.2, p. 224]. Here two disjoint edges cannot even share an endpoint. This result shows that Conway's Thrackle conjecture holds for "linear thrackles" [O'R95].

10. Any set of n points in the plane in general position has a spanning path such that any line intersects $O(\sqrt{n})$ segments of the path [Thm. 15.12 (Welzl), p. 258; Ex. 15.13, p. 264].

11. For any two-coloring of the integer points (x, y) with $1 \leq x, y \leq \sqrt{n}$, there is a halfplane within which one of the colors outnumbers the other by $\Omega(n^{1/4})$ [Thm. 16.16 (Alexander), p. 285].

References

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