Computational Geometry Column 29

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Abstract

Past accomplishments of Computational Geometry are reviewed and future directions adumbrated.

Computational Geometry is by some measures twenty years old, old enough to warrant a celebration of accomplishments as well as a glance toward the future.

Past Accomplishments

1. *Simple polygons.* For nearly every problem based on simple polygons, asymptotically optimal algorithms have been found. One of the last to succumb was triangulation: a simple polygon of n vertices can be triangulated in $O(n)$ time [Cha91].

2. *Visibility graphs.* Efficient computation of visibility graphs has been achieved in a polygon and in a polygonal environment: for graphs of k edges, $O(n+k)$ time suffices to find all shortest path trees (from a vertex to all others) in a polygon [GHL+87] [Her89], and $O(n \log n + k)$ is achievable for construction of the visibility graph among obstacles in the plane [GM91].

3. *Segment intersection.* After many attempts, an output-size optimal algorithm was constructed for intersecting n segments: in $O(n \log n + k)$ time, where k is the number of pairs of intersecting segments [CE92] [Bal95].

4. *Convex Hulls.* The convex hull of n points in d dimensions has $O(n^{\lfloor d/2 \rfloor})$ facets, and asymptotically worst-case optimal $O(n \log n + n^{\lfloor d/2 \rfloor})$ algorithms were found both for even [Sei81] and odd [Cha93] dimensions.

5. *Voronoi diagrams and Delaunay triangulations.* An optimal and practical sweepline algorithm was discovered for constructing Voronoi diagrams in the plane [For87]; implementations are now widely distributed. The deep insight that Delaunay triangulations, duals of Voronoi diagrams, are projections of convex hulls from one higher dimension [Bro79] [ES86], unified two principal lines of research.

6. *Complexity of Arrangements.* Great strides were made in establishing the complexity of arrangements:

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Supported by NSF grant CCR-9421670.

(a) The "Zone Theorem" established that the "neighborhood" of any one hyperplane in an arrangement of n hyperplanes in d-dimensions has complexity $O(n^{d-1})$ [ESS93].

(b) A difficult technical achievement was showing that any m faces in an arrangement of n lines in the plane have total complexity $O(m^{2/3}n^{2/3}+m+n)$ [CEG+90]. This bound applies, for example, to the number of incidences between n points and m lines.

(c) One long-sought result is that the complexity of the lower envelope of n surface patches in d-dimensions is $O(n^{d-1+\epsilon})$ (for any $\epsilon > 0$) [Sha94].

(d) The beautiful and intricate theory of Davenport-Schinzel sequences was shown to be central to the combinatorics of arrangements [AS95], establishing, for example, that the complexity of the lower envelope of n (perhaps interpenetrating) segments in the plane is $\Theta(n\alpha(n))$, where α is the inverse Ackermann function.

7. *Motion planning.* Two general algorithms for solving any motion planning problem were developed: cell decomposition [SS83] and the roadmap algorithm [Can87], both exponential in the number of degrees of freedom of the robot. For many special cases more efficient algorithms were found, most notably for a polygon translating and rotating in the plane, for which a nearly-quadratic algorithm was developed [HS93].

8. *Parallel Algorithms.* Optimal or near-optimal work bounds were realized for many geometry problems under a variety of parallel computing models. For example, algorithms for the convex hull (cf. item 4 above) achieve in the EREW PRAM model an optimal runtime-processors product of $O(n^{\lfloor d/2 \rfloor})$ for even d, and a polylog(n) factor more for odd d [AGR94].

9. *Linear Programming.* Remarkably, it was established that linear programming could be accomplished in linear time (in the number of constraints) for fixed dimension d [Meg84]. The doubly-exponential dependence on d was steadily improved [Cla88], and eventually simple randomized algorithms were found whose expected dependence on d is subexponential [MSW92] [Kal92]. These results generalize to other optimization problems, such as finding the minimum spanning ellipsoid.

10. *Range searching.* A significant achievement in the last decade was the near-complete resolution of the range searching problem, with almost-matching upper and lower bounds. The introduction of ε-nets [HW87] led to linear-size data structures with near-optimal query times for simplex range searching (reporting points inside query simplices). Trading off space with query time permits achieving faster (logarithmic) query times, using "$1/r$-cuttings": given an arrangement of n lines and a parameter $r < n$, the plane may be quickly (and deterministically) decomposed into $O(r^2)$ "triangles" (some unbounded), so that no triangle meets more than $O(n/r)$ of the lines [Cla87] [Mat90] [Aga90]. A similar result holds for arrangements of hyperplanes in d dimensions [CF90].

11. *Randomized geometric algorithms.* The introduction of random sampling techniques [Cla87] revitalized the field by showing that many complex geometric prob-

lems have startlingly simple and efficient randomized solutions. Notable is the simple randomized incremental algorithm for construction of the convex hull [CS89], whose expected running time in d dimensions is asymptotically optimal. Simple optimal expected-time algorithms were also found for segment intersection (item 3 above) [Mul88] [CS89]. Derandomization led to several advances in deterministic running times, the even-d optimal hull algorithm (item 4 above) being a prominent instance. Finally, the introduction of the "backwards analysis" technique led to running-time analyses for randomized algorithms as simple as the algorithms themselves [Sei93].

Future Directions

Between the release of the "CG Impact Task Force Report" [Cha96] and a recent ACM Workshop on Strategic Directions in Computing Research (held at MIT June 14-15, 1996), the computational geometry community has been in an unusually reflective mood. The discussions make it possible to discern possible future directions for the field.

One might crudely characterize the space of geometry problems so far explored by two parameters: the dimension of the geometric space, and the shape complexity of the objects in that space. Most of our accomplishments fall into the low-dimension, non-complex corner of this space: $d = 2$ or 3, and points, segments, lines, polygons, or planes; or fixed dimension d, and convex polytopes or arrangements of hyperplanes. Higher dimensions (e.g., $d = 10^4$) and curved, non-convex geometric objects are relatively unexplored by the computational geometry community.

Earlier work also may be characterized by the methodology used by researchers to approach the problems. Most of the accomplishments listed above resulted from the pursuit of asymptotically-optimal algorithms. Greater emphasis is now being placed on computationally-robust algorithms, on approximations, and on the need for simple, practical algorithms. Treating "methodology" as a third parameter, we can view our accomplishments as lying primarily in the asymptotically-optimal slice of the three-parameter space (and in the small d, low-complexity corner of that slice). A vast region remains to be explored.

Acknowledgements. The problem-space view of accomplishments just expressed was sketched by the sixteen members of the Computational Geometry Working Group of the ACM Workshop on Strategic Directions in Computing Research. I also benefited from comments by K. Clarkson, S. Fortune, J. E. Goodman, J. Hershberger, J. Mitchell, R. Seidel, and E. Welzl. Remaining distortions, omissions, and other flaws are mine.

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(Written 30 July 1996)