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Note

Each Four-Celled Animal Tiles the Plane

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An *animal* A is a set of unit squares in the plane, parallel to the axes, and with corners at integer lattice points. We show that any animal A with four cells *tiles* the plane, in the sense that infinitely many copies of A, translated by integer vectors and possibly rotated through 90°, 180°, or 270°, can be placed so as to fill the plane exactly without overlap.

The following problem was published by Klarner [1]: If a four-celled animal A consists of four unit squares, aligned on the integer lattice though not necessarily connected, can copies of A, translated and rotated, tile the plane? We give an algorithm to show that they can. We leave open the similar question for five-celled animals. For six or more cells, the answer is generally "no," while for three or fewer it is always "yes."

The problem is a special case of the class of questions studied earlier (but published later) by Gordon [2]: Let S be a finite subset of Z^n. Let G_n be the group of isometries of R^n and H a subgroup of G_n. What conditions on n, H, and $|S|$ ensure that copies of S, translated by different elements of H, disjointly tile Z^n (or some prescribed subset T of Z^n)?

DEFINITION. An *animal* A is a set of unit squares in the plane, parallel to the axes, with corners at integer lattice points. An animal need not be connected.

DEFINITION. An animal A *tiles the plane* if we can place infinitely many copies of A, translated by integer vectors and possibly rotated through 90°, 180°, or 270°, and fill the plane exactly without overlap. (We do not count overlapping edges or corners.)

Remark. We may also consider the animal to be set of lower left-hand corners of squares. In this sense, tiling the plane is equivalent to packing

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the integer lattice. We will refer to the lower left-hand corners of the squares of A as the *points* of A.

We will show that each four-celled animal, A tiles the plane, answering the question cited in [1]. We will outline the construction, and leave many of the details to the reader.

Notation. "$2^k \mid\mid m$" (read "2^k exactly divides m") means that 2^k divides m but no higher power of 2 will divide m. "A is more even than B" means that the power of 2 which divides A is strictly greater than that which divides B.

THEOREM. *Any 4-celled animal A tiles the plane.*

Proof. We break into several cases:

(1) Exactly two points of A lie on a given horizontal or vertical line, and the other two points are not joined by a parallel line.

(2) All four points lie in the same horizontal (vertical) line.

(3) Exactly two points lie on a given skew line, and the other two points are not joined by a parallel line.

(4) All four points lie in the same skew line.

These cases are exhaustive, although not mutually exclusive: an animal could satisfy both 1. and 3.

Let us consider these cases separately.

Case 1. Two points of A are on a horizontal line, the other two are not on this line, and the other two are not joined by a horizontal line. Let the points of A be given as $(0, 0)$, $(b, 0)$, (c, e), (d, f), without loss of generality. Replicate the animal b times with horizontal step 1. (That is, lay down b copies of A, each removed from the previous by a horizontal step of 1, i.e., the vector $(1, 0)$. Implicit in the "replication" is the assertion that the copies do not overlap; this is an exercise for the reader each time.) Then replicate the entire pattern infinitely often with horizontal step 2b. This fills the line $y=0$, and half-fills the lines $y=e$ and $y=f$ in a regular manner (b filled spots, then b empty spots, alternating). Then break into subcases.

Case 1.1. $(f-e)$ is more even than either f or e. Say $2^m \mid\mid f-e$. Then replicate the pattern infinitely often with steps of $(x, 2^m)$, where x is chosen to mesh the half-filled lines when necessary. We then have a pattern of filled horizontal lines at each $y=0 \pmod{2^m}$ and $y=e \pmod{2^m}$, and emptiness elsewhere. Also, $e \neq 0 \pmod{2^m}$. If $2^p \mid\mid e$, we then replicate this pattern 2^p times with vertical step 1, and finally replicate the resulting pattern 2^{m-p-1} times with vertical step 2^{p+1}. This exactly fills the plane.

Case 1.2. e is more even than $(f-e)$. (If f is more even, just rename the boxes). We rotate through 180° the pattern of one filled and two half-filled horizontal lines, translate, and merge with the original to exactly fill the lines $y=0$, $y=e$, $y=f$, $y=e+f$. (The same translation which meshes the two half-lines at $y=e$ to fill the line, also meshes the two half-lines at $y=f$.) Now suppose $2^p \| (f-e)$ and $2^m \| e$. Replicate the pattern 2^{m-p-1} times with vertical step 2^{p+1}, then replicate the result infinitely often with vertical step 2^{m+1}. This produces a pattern of filled horizontal lines at each $y=0 \pmod{2^p}$ and emptiness elsewhere. Replicate 2^p times with vertical step 1 to tile the plane.

(Note: Since e, f, and $e-f$ are all nonzero, one is more even than the other two, so that Cases 1.1 and 1.2 exhaust Case 1.)

Case 2. All four points lie on the same horizontal line. Let their coordinates be $(0,0)$, $(b,0)$, $(c,0)$, $(d,0)$. Assume $\gcd(b,c,d)=1$, since otherwise we could divide all coordinates by $\gcd(b,c,d)$, tile the plane, enlarge the picture by a scale of $\gcd(b,c,d)$, and replicate the resulting pattern vertically $\gcd(b,c,d)$ times with step 1, then horizontally $\gcd(b,c,d)$ times with step 1.

Case 2.1. 0, b, c, d are all different $(mod\, 4)$. We simply replicate the pattern infinitely often with horizontal step 4 to tile the line, then replicate infinitely often with vertical step 1 to tile the plane.

Case 2.2. 0, b, c, d are all different $(mod\, 8)$, *but two are the same* $(mod\, 4)$. We can assume two coordinates are 0 and 4 (mod 8). There must be at least one odd coordinate, which we assume to be 1 (mod 8) without loss of generality. (We may have used translation and/or rotation through 180° to achieve this.) The fourth coordinate may be either 2, 3, 5, 6, or 7 (mod 8). In any event, replicate the pattern infinitely often with horizontal step 8. Then proceed according to the values of the four coordinates (mod 8):

(0, 1, 2, 4) Rotate 180° and translate to yield (7, 6, 5, 3), then mesh with the original to tile the line.

(0, 1, 3, 4) Consider Fig. 1, repeated throughout the plane, with horizontal and vertical periods of 8 each. Use this as a template, with the X's indicating elements of horizontal lines and the 0's elements of vertical lines. This template shows how to weave horizontal lines (the pattern) and vertical lines (the pattern rotated through 90°) to tile the plane.

(0, 1, 4, 5) Translate 2 to the right to yield (2, 3, 6,7); mesh with the original to tile the line.

(0, 1, 4, 6) Rotate through 180° and translate to yield (3, 2, 7, 5); mesh with the original to tile the line.

```
0 0 0 X X 0 X X
0 X X 0 0 0 X X
X X 0 0 0 X X 0
X X 0 X X 0 0 0
X 0 X X 0 0 0 X
0 0 X X 0 X X 0
0 X X 0 X X 0 0
X 0 0 0 X X 0 X
```

FIGURE 1

(0, 1, 4, 7) Use Fig. 1 again, but with 0's representing elements of horizontal lines and X's elements of vertical lines.

Case 2.3. Two abscissas are equivalent (mod 8). Let the most even of the differences between abscissas be exactly divisible by 2^k, and the next most even difference, by 2^n. (We may have $n = k$). k is at least 3 and n is at least 1. The least even differences are in fact odd, and there are three or four of these.

Case 2.3.1. $k = n$. Then we have two even abscissas, whose difference is exactly divisible by 2^k, and two odd abscissas, whose difference is also exactly divisible by 2^k, Replicate horizontally 2^{k-1} times with a step of 2, then replicate infinitely often with a horizontal step of 2^{k+1} to tile the line.

Case 2.3.2. $k > n$, *and we have three even and one odd abscissas* (*or three odd and one even*). Rotate through 180° and translate, and combine with the original pattern, so that we obtain two patterns, one on the even points of the line and one on the odd points, both being displacements of the pattern $(0, e, f, e+f)$, with e more even than $(e-f)$. This is always possible, although e and/or f may be negative. We do know that e, f, $e+f$, and $e-f$ are all nonzero. Now proceed as in Case 1.2: if $2^p \| (f-e)$ and $2^m \| e$, we replicate the pattern 2^{m-p-1} times with horizontal step 2^{p+1}, then replicate the result infinitely often with horizontal step 2^{m+1}. This produces a pattern of filled cells in each of two arithmetic progressions, each with pitch 2^p and displaced an odd amount from each other. Replicate the result 2^{p-1} times with horizontal step 2 to tile the line.

Case 2.3.3. $k > n = 1$, *two even and two odd abscissas.* We replicate infinitely often with horizontal step 2^{k+1} to get a pattern, periodic with period 2^{k+1}. The "full cycle" (any interval of size 2^{k+1}) consists of two cells spaced 2^k apart, and two other cells spaced $(4q+2)$ apart for some integer q, with the two pairs of cells at odd displacement from one another. Set $m = k$, skip the next paragraph, and proceed.

Case 2.3.4. $k > n > 1$, *two even and two odd abscissas.* We can obtain the same pattern as Case 2.3.3. First replicate the pattern 2^{k-n-1} times with horizontal step 2^{n+1}, then replicate the result infinitely often with horizontal step 2^{k+1}. This gives an arithmetic progression A of points, with pitch 2^{n+1}, and two sequences B and C of cells, each occupying exactly half of the elements of a progression of pitch 2^{n+1} by alternately occupying and emptying strings of 2^{k-n} within that progression. The progressions thus half-occupied are removed from each other by 2^n, and removed from A with an odd displacement. Now by rotating this pattern 180°, and translating, we get a new progression A' and two half-progressions B' and C'. We can combine with the original pattern in such a way that B and C' merge to form a complete progression, as do B' and C, while A' is removed from A by a number exactly divisible by 2. That is, we have obtained a pattern of period 2^{n+1} whose full cycle consists of two cells at distance 2^n apart, two other cells at distance $(4q+2)$ apart for some integer q, and the two pairs of cells displaced from each other by odd distance. Set $m=n$, and rejoin Case 2.3.3.

Cases 2.3.3 *and* 2.3.4 *combined*: Replicate the pattern 2^m times with horizontal step 4. Now we have filled everything equivalent to 0 (mod 4), half of the positions equivalent to 1 (mod 4), and half of those equivalent to 3 (mod 4), in such a way that an odd-numbered cell is covered if and only if the cell 2^m away is not covered. In the plane, lay down a copy of this pattern with translation $(2p, 2p)$ for each integer p. Then rotate the pattern through 90° and lay down a copy with translation $(2p+1, 2p+1-2^m)$ for each integer p. This fills in a pattern such as in that in Fig. 2. (The cell marked X' is at $(0, 0)$, each X is covered, and each "." is not.) Finally, translate the entire pattern with step $(-1, 1)$ and merge with the original pattern to cover the plane.

Case 3. *Exactly two points lie on a given skew line, and the other two points neither lie on this line nor determine a parallel line.* Let the skew line have slope p/q, with p and q relatively prime. There are integers m and n such that $mp+nq=1$. Transform the points of A (the coordinates of the lower left-hand corners of its cells) according to the linear transformation $f(x, y)=(nx+my, -px+qy)$. This transformation has determinant 1, so that it maps the integer lattice one-one onto itself. In particular, it maps the collection of lower left-hand corners of squares of A into the integer lattice,

```
X...X...X...X...X...
X.XXX.XXX.XXX.XXX.XX
..X...X...X...X...X.
XXX.XXX.XX'X.XXX.XXX.
X...X...X...X...X...
X.XXX.XXX.XXX.XXX.XX
..X...X...X...X...X.
XXX.XXX.XXX.XXX.XXX.
```

FIG. 2. A half-filled plane

defining a new animal with two cells on a horizontal line and the other two cells neither lying on that line nor determining a parallel line. Use Case 1 to tile the plane with this new animal. Note that Case 1 only uses the operations of translation and 180° rotation, which are preserved in any linear transformation. Finally, applying the inverse transformation $f^{-1}(x', y') = (qx' - my', px' + ny')$ to the converted integer lattice, we get back a covering of the integer lattice (since the reverse transformation is also one to one and onto) with the original animal.

Case 4. *All four points lie on a skew line.* Say their coordinates are $(0, 0)$, (bp, bq), (cp, cq), (dp, dq), with p and q relatively prime. Use Case 2 to cover an integer lattice with animals of shape $[(0, 0), (b, 0), (c, 0), (d, 0)]$. This transformation amounts to a strict rotation and an expansion by a factor of $\sqrt{p^2 + q^2}$. Thus it preserves all operations we used in Case 2: translations and rotations through 90°, 180°, or 270°. So we have a covering of that subset of the integer lattice for which $px + qy = 0 \pmod{p^2 + q^2}$, using valid operations on the original animal. Finally, replicate this pattern $p^2 + q^2$ times, with horizontal step 1, to cover the plane. Q.E.D.

Generalizations: The six-celled animal

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XX
X X
 XX
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cannot possibly cover the plane, even allowing reflections. [1, 2] Can every five-celled animal cover the plane?

More generally, Gordon [2] gives sets S of size $3n$ when n is even, or of size $3n + 1$ when n is odd, such that S cannot tile Z^n under G_n. We note that for arbitrary $n \geqslant 2$, the following set S, of size $3n$, also fails to tile Z^n under the group G_n: S consists of the $2n$ points $\pm(1, 0, \ldots, 0)$, $\pm(0, 1, \ldots, 0), \ldots, \pm(0, 0, \ldots, 1)$, the $n - 1$ points $(1, -1, 0, 0, \ldots, 0, 0, 0)$, $(0, 1, -1, 0, \ldots, 0, 0, 0), \ldots, (0, 0, 0, 0, \ldots, 0, 1, -1)$, and the point $(-1, 0, \ldots, 0, 1)$.

Are these the smallest such examples? Let $f(n)$ be the size of the smallest subset S of Z^n which fails to tile Z^n. What is $f(n)$? Also, Gordon [3] asks: For each $m \geqslant f(n)$, are there sets S which of size m which fail to cover Z^n? Is $f(n)$ nondecreasing? These questions are left open.

References

1. V. Chvatal, D. A. Klarner, and D. E. Knuth, "Selected Combinatorial Research Problems," Report No. STAN-CS-72-292, Problem No. 9, Stanford University, June 1972.
2. B. Gordon, Tilings of lattice points in Euclidean n-space, *Discrete Math.* **29** (1980), 169–174.
3. B. Gordon, private communication.